SOLUTION OF FRACTIONAL ORDER DIFFUSION EQUATION WITH NON-LINEAR TERM BY MODIFIED DECOMPOSITION METHOD

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ABSTRACT: *In this paper a general solution of space bounded one dimensional fractional diffusion equation with a non-linear term is given. Modified Adomian Decomposition Method is used to express the function y (x , t) which is to be found as an infinite series and the non-linear term is decomposed in to infinite series of polynomials. Riemann Liouville derivative is used for time fractional derivative. Two examples are presented at the end. The comparisons of analytical and approximate solutions along with graphical comparisons for different values of "t" exhibits the utility of MDM.*

Keywords: Fractional derivative, fractional order diffusion equation, Riemann Liouville derivative, one dimensional diffusion equation with non-linear term, space bounded equations, modified decomposition method (MDM).

1. INTRODUCTION

The branch of mathematics namely fractional calculus has become popular among mathematicians in the last four decades due to its diverse and widespread approach in different fields of physical sciences. It not only provides a handful of significant techniques for solving differintegral equations of arbitrary order but also provide useful methods for problems of special functions of mathematical physics[1]. Fractional order differintegral equations are highly economical and useful to formulate the specific electro chemical problems as compare to classical techniques in terms of Fick's laws of diffusion [2]. Fractional calculus makes contact with a very large segment of classical analysis and provides a unifying theme for a great many known and some new results. Its applications outside mathematics includes transmission line theory, chemical analysis of aqueous solutions, design of heat flux meters, rheology of soils, growth of inter granular grooves at metal surfaces, quantum mechanical calculations, and dissemination of atmospheric pollutants. In the last few years' scholars show keen interest in fractional order diffusion equations due to their vast ability to handle the sub diffusive and super diffusive processes. Diffusive transport in semi-infinite medium is the most powerful application of fractional calculus. Now a day's diffusion equations are used to process the fractional systems and their signals in control theory[3]. Fraction diffusive equations also used to establish the anomalous dispersion models[4] and has the remarkable implementation in mathematical physics[5]. In this paper, the numerical solution of space fractional diffusion equation with nonlinear source term by "Modified Decomposition Method" (MDM)[6] is presented. It presented an extensive interpretation of the numerical solution of the partial differintegral equations of arbitrary order. It will provide the critics an exhaustive analysis of numerical methods for partial differintegral equations of arbitrary order available today. Moreover, it indicates the facts which have to be persuaded with care in their imposition. Numerical examples are presented to validate the method.

2. METHOD DESCRIPTION

Consider the general nonlinear equation

$$m(y) + q(y) + g(y) = h(t) \quad (1)$$

Where "m" represent the highest ordered differential operator with respect to time and "q" is the remainder of the linear operator. The source term is illustrated by g(y) [6]. Now we have

$$m(y) = h(t) - q(y) - g(y) \quad (2)$$

Let m^{-1} is defined as an integral operator as

$$m^{-1} = \int_0^t (.)\, dt \quad (3)$$

Apply M^{-1} on both side of (2), we get

$$y = g_0 + m^{-1}\{h(t)-q(y)-g(y)\} \quad (4)$$

Where g_0 is the homogeneous equation

$$m(y) = 0 \quad (5)$$

We find the constants with help of given conditions (initial or boundary), depending on the nature of the problem whether it is an initial value or boundary value problem respectively. Adomian's approach was that the untold function y(r, t) can be expressed in terms of an infinite series as follows

$$y(x,t) = \sum_{n=0}^{n=\infty} y_n(r,t) \quad (6)$$

and the non-linear operator g(y) can be split into an infinite series of polynomial as follows

$$g(y) = \sum_{n=0}^{n=\infty} b_n \quad (7)$$

Where y_n (r, t) will be determined reiteratively, and "b_n" are the polynomials of $y_1, y_2, y_3, \ldots, y_n$ given by

$$b_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[g\left(\sum_{n=0}^{n=\infty}\lambda^i y_i\right)\right]_{\lambda=0} \quad (8)$$

MDM was introduced by Wazwaz [7]and it is depending on the hypothesis that the function g(x) can be splits into two parts, namely $g_0(r)$ and $g_1(r)$ under this hypothesis we get

$$g(r) = g_0(r) + g_1(r) \quad (9)$$

This approach is applying when the function g consists of several terms and can be split into two different parts. In this case, g is usually a summation of any one of polynomial and trigonometric or transcendental function. A suitable selection of the function g_0 (r) is important. If g_0 is made up of one term of “g” or minimum number of terms if possible and g_2 comprise of the rest of the terms of “g” the method would be more productive.

2. APPLICATION OF MDM ON ONE DIMENSIONAL SPACE FRACTIONAL DIFFUSION EQUATION

To solve one dimensional space fractional diffusion equation with source term we apply MDM [8] to

$$\frac{\partial y(r,t)}{\partial t}=q(r)\frac{\partial^{\alpha}y(r,t)}{\partial r^{\alpha}}+U(r,t) \quad (10)$$

Suppose that

$$y=\sum_{n=0}^{n=\infty} y_n \quad (11)$$

be the solution of (1), so (1) becomes

$$m_t y(r,t)=q(r)\ D_r^{\mu}(r,t)+U(r,t) \quad (12)$$

Where $m_t=\frac{\partial}{\partial t}$ which is simply invertible operator, $D_r^{\propto}(.)$ is the Reimennan liouville Derivative of order ∝. Now by modified decomposition method, we have

$$y(r,t)=y(r,0)+m_t^{-1}\left\{p(r)D_r^{\alpha}\left(\sum_{n=0}^{n=\infty}y_n\right)\right\}+L_t^{-1}(U(r,t)) \quad (13)$$

each term of infinite series (7) is given by recurrence relation of MDM

$$y_0=g_0 \quad (14)$$

$$y_1=g_1+m_t^{-1}\left\{q(r)D_r^{\alpha}y_0\right\} \quad (15)$$

$$y_{n+1}=m_t^{-1}\left\{p(r)D_r^{\alpha}y_n\right\} \quad (16)$$

Where $n \geq 0$

When y_0 is defined then the remaining constituents of y_n, where $n \geq 1$ can entirely be resolved and every term is obtained by using the preceding term. Consequently, the computed $y_1, y_2, \dots$ are defined and the series solutions thus completely obtained. Nevertheless, in numerous cases the solution obtained by MDM is in a close agreement to the exact solution.

3.1 Numerical Examples

Example 1

Consider the space fractional diffusion equation [9]

$$\frac{\partial y(r,t)}{\partial t}=q(r)\frac{\partial^{1.8}y(r,t)}{\partial r^{1.8}}+U(r,t) \quad (17)$$

on a bounded domain $0 < r < 1$ with the diffusion coefficient [10]

$$q(r)=\Gamma(2.2)\frac{r^{2.8}}{6}=0.183634r^{2.8} \quad (18)$$

$$U(r,t)=-r^3(1+r)e^{-t} \quad (19)$$

with the initial condition

$$y(r,0)=r^3 \quad \text{for } 0<r<1 \quad (20)$$

and the boundary conditions

$$y(0,t)=0,\ y(1,t)=e^{-t} \quad (21)$$

Solution:

We can re write Equation (17) as

$$m_t y(r,t)=q(r)\ D_r^{1.8}y(r,t)+U(r,t) \quad (22)$$

Where $m_t=\frac{\partial}{\partial t}$ symbolizes the simple invertible linear differential operator. $D_r^{1.8}(.)$ is the Riemann- Liouville derivative of order 1.8. If the invertible operator $m^{-1}=\int_0^t(.)dt$ is applied to equation (22) then we get

$$m_t^{-1}m_t y(r,t)=m_t^{-1}\left\{q(r)\ D_r^{1.8}y(r,t)+U(r,t)\right\} \quad (23)$$

On simplifying equation (23)

$$y(r,t)-y(r,0)=m_t^{-1}\left(q(r)D_r^{1.8}y(r,t)+U(r,t)\right)$$

$$y(r,t)=y(r,0)+m_t^{-1}\left(P(r)D_r^{1.8}y(r,t)\right)+L_t^{-1}\left(U(r,t)\right) \quad (24)$$

$$y(r,t)=r^3+m_t^{-1}\left(q(r)D_r^{1.8}y(r,t)\right)-(1+r)r^3\int_0^t e^{-t}dt$$

$$=r^3+m_t^{-1}\left(q(r)D_r^{1.8}y(r,t)\right)+(1+r)r^3(e^{-t}-1)$$

$$=r^3+m_t^{-1}\left[q(r)D_r^{1.8}y(r,t)\right]+(r^3+r^4)e^{-t}-r^3-r^4$$

is determined. The key point is that the solution of (17) by MDM is in the form

$$y(r,t)=\sum_{n=0}^{n=\infty}y_n(r,t) \quad (25)$$

While solving we approximate the solution as follows

$$g(r,t)=r^3+(r^3+r^4)e^{-t}-r^3-r^4 \quad (26)$$

$$g_0=r^3 \quad (27)$$

$$g_1=-(r^3+r^4)(1-e^{-t}) \quad (28)$$

$$y_1(r,t)=g_1+m_t^{-1}\left(d(r)D_r^{1.8}y_0(r,t)\right)$$

$$y_1(r,t)=(r^3+r^4)(e^{-t}-1)+m_t^{-1}\left(q(r)D_r^{1.8}r^3\right)$$

$$=(r^3+r^4)(e^{-t}-1)+\frac{0.183634\Gamma(4)}{\Gamma(2.2)}r^{2.8}\frac{\Gamma(3+1)}{\Gamma(3+1-1.8)}r^{3-1.8}$$

$$=(r^3+r^4)(e^{-t}-1)+0.183634\frac{\Gamma(4)}{\Gamma(2.2)}r^4t$$

and after successive iterations we have

$$y_2(r,t)=(1-t-e^{-t})\left[\frac{\Gamma(4)r^4}{\Gamma(2.2)}+\frac{\Gamma(5)r^5}{\Gamma(3.2)}\right]+\left[\frac{\Gamma(4)\Gamma(5)}{\Gamma(2.2)\Gamma(3.2)}\right]r^5t$$

3.2 COMPARISON OF ANALYTICAL AND MODIFIED DECOMPOSITION METHOD

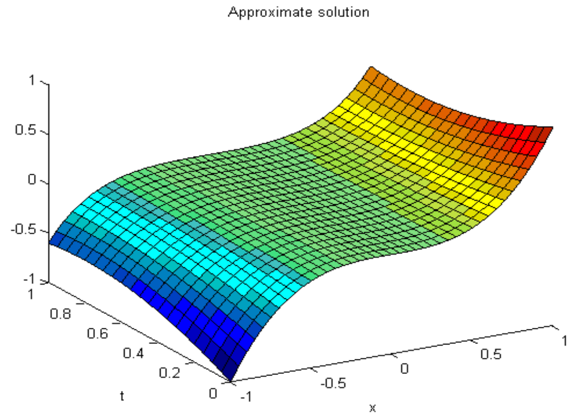

Figure-1 Solution by Modified Decomposition Method

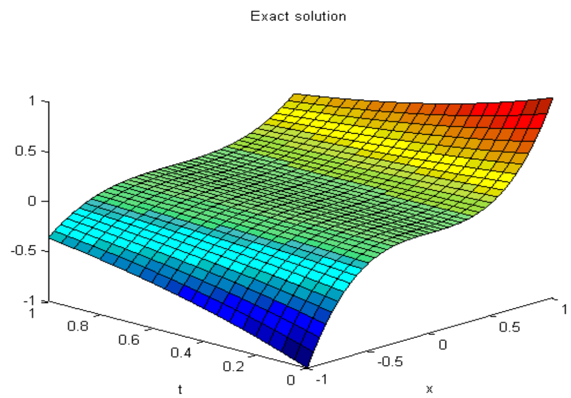

Figure-2 Exact solution

Table-1 When t = 0.1

x	t	Approximate	Exact	Error
0.00000	0.10000	0.00000	0.00000	0.00000
0.05000	0.10000	0.00011	0.00011	0.00000
0.10000	0.10000	0.00090	0.00090	0.00000
0.15000	0.10000	0.00305	0.00305	0.00000
0.20000	0.10000	0.00723	0.00724	0.00000
0.25000	0.10000	0.01413	0.01414	0.00000
0.30000	0.10000	0.02443	0.02443	0.00000
0.35000	0.10000	0.03879	0.03879	0.00000
0.40000	0.10000	0.05791	0.05791	0.00000
0.45000	0.10000	0.08245	0.08245	0.00001
0.50000	0.10000	0.11311	0.11310	0.00001

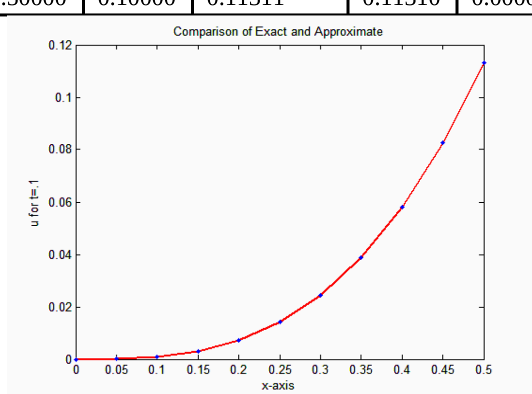

Figure-3 When t = 0.1

Table-2 When t = 0.15

x	t	Approximate	Exact	Error
0.00000	0.15000	0.00000	0.00000	0.00000
0.05000	0.15000	0.00011	0.00011	0.00000
0.10000	0.15000	0.00086	0.00086	0.00000
0.15000	0.15000	0.00290	0.00290	0.00000
0.20000	0.15000	0.00689	0.00689	0.00000
0.25000	0.15000	0.01345	0.01345	0.00000
0.30000	0.15000	0.02324	0.02324	0.00000
0.35000	0.15000	0.03691	0.03690	0.00001
0.40000	0.15000	0.05510	0.05509	0.00001
0.45000	0.15000	0.07845	0.07843	0.00002
0.50000	0.15000	0.10762	0.10759	0.00003

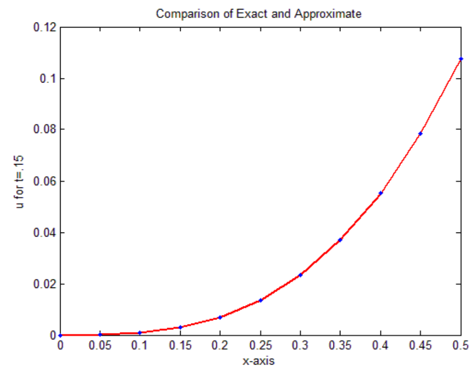

Figure-4 When t = 0.15

Table-3 When t = 0.2

x	t	Approximate	Exact	Error
0.00000	0.20000	0.00000	0.00000	0.00000
0.05000	0.20000	0.00010	0.00010	0.00000
0.10000	0.20000	0.00082	0.00082	0.00000
0.15000	0.20000	0.00276	0.00276	0.00000
0.20000	0.20000	0.00655	0.00655	0.00000
0.25000	0.20000	0.01279	0.01279	0.00000
0.30000	0.20000	0.02211	0.02211	0.00001
0.35000	0.20000	0.03512	0.03510	0.00001
0.40000	0.20000	0.05242	0.05240	0.00002
0.45000	0.20000	0.07465	0.07461	0.00004
0.50000	0.20000	0.10241	0.10234	0.00007

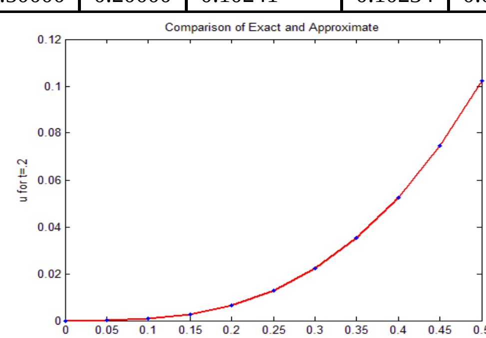

Figure-5 When t = 0.2

Table-4 When t = 0.25

x	t	Approximate	Exact	Error
0.00000	0.25000	0.00000	0.00000	0.00000
0.05000	0.25000	0.00010	0.00010	0.00000
0.10000	0.25000	0.00078	0.00078	0.00000
0.15000	0.25000	0.00263	0.00263	0.00000
0.20000	0.25000	0.00623	0.00623	0.00000
0.25000	0.25000	0.01217	0.01217	0.00000
0.30000	0.25000	0.02104	0.02103	0.00001
0.35000	0.25000	0.03341	0.03339	0.00002
0.40000	0.25000	0.04989	0.04984	0.00005
0.45000	0.25000	0.07105	0.07097	0.00008
0.50000	0.25000	0.09749	0.09735	0.00014

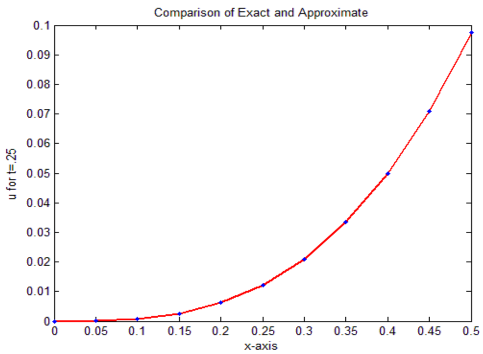

Figure-6 When t = 0.25

Table-5 When t = 0.3

x	t	Approximate	Exact	Error
0.00000	0.30000	0.00000	0.00000	0.00000
0.05000	0.30000	0.00009	0.00009	0.00000
0.10000	0.30000	0.00074	0.00074	0.00000
0.15000	0.30000	0.00250	0.00250	0.00000
0.20000	0.30000	0.00593	0.00593	0.00000
0.25000	0.30000	0.01158	0.01158	0.00001
0.30000	0.30000	0.02002	0.02000	0.00002
0.35000	0.30000	0.03180	0.03176	0.00004
0.40000	0.30000	0.04749	0.04741	0.00008
0.45000	0.30000	0.06765	0.06751	0.00014
0.50000	0.30000	0.09284	0.09260	0.00024

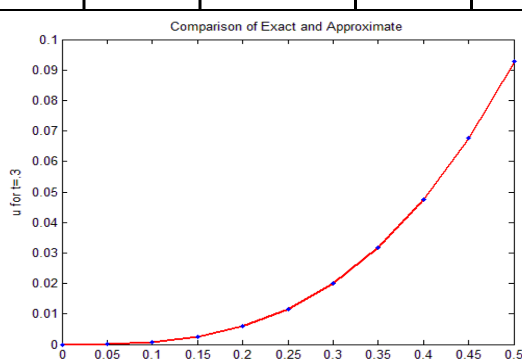

Figure-7 When t = 0.3

Example 2

Consider the space fractional diffusion equation

$$\frac{\partial y(r,t)}{\partial t}=q(r)\frac{\partial^{1.8}y(r,t)}{\partial r^{1.8}}+U(r,t) \qquad (29)$$

on a bounded domain, 0 < r < 1 with the diffusion coefficient

$$q(x)=\Gamma(1.2)x^{1.8}\partial \qquad (30)$$

$$V(r,t)=3r^2(2r-1)e^{-t} \qquad (31)$$

with the initial and boundary conditions

$$y(r,0)=r^2-r^3 \qquad (32)$$

$$y(0,t)=0, \quad y(1,t)=0 \qquad t>0 \qquad (33)$$

respectively.

Solution

We re write equation (29) as

$$m_t y(r,t)=q(r)\ D_x^{1.8}(r,t)+V(r,t) \quad (34)$$

Where $m_t=\frac{\partial}{\partial t}$ is a simple invertible linear differential operator. $D_r^{1.8}(.)$ is the Riemann- Liouville derivative of order 1.8. By applying invertible operator $m^{-1}=\int_0^t(.)dt$ to equation (34) we will get

$$m_t^{-1}m_t y(r,t)=m_t^{-1}\left(q(r)\ D_r^{1.8}(r,t)+V(r,t)\right) \qquad (35)$$

On simplifying (35), we have

$$y(r,t)-y(r,0)=m_t^{-1}\left(q(r)D_r^{1.8}(r,t)+V(r,t)\right)$$

$$y(r,t)=y(r,0)+m_t^{-1}\left(q(x)D_r^{1.8}(r,t)\right)+m_t^{-1}\left(V(r,t)\right) \qquad (36)$$

$$y(r,t)=r^2-r^3+m_t^{-1}\left(q(r)D_r^{1.8}(r,t)\right)-m_t^{-1}\left(3r^2(2r-1)e^{-t}\right)$$

$$y(r,t)=r^2-r^3+m_t^{-1}\left(q(r)D_r^{1.8}(r,t)\right)-3r^2(2r-1)\int_0^t e^{-t}dt \qquad (37)$$

$$y(r,t)=r^2-r^3+m_t^{-1}\left(q(r)D_r^{1.8}(r,t)\right)-(6r^3-3r^2)(e^{-t}-1)$$

We suppose

$$g_0=(r^2-r^3) \qquad (38)$$

$$g_1=(6r^3-3r^2)(1-e^{-t}) \qquad (39)$$

$$y_1(r,t)=g_1+m_t^{-1}\left(q(r)D_r^{1.8}y_0(r,t)\right)$$

$$=(6r^3-3r^2)(1-e^{-t})_1+L_t^{-1}\left[\Gamma(1.2)r^{1.8}\left(D_x^{1.8}(r^2-r^3)\right)\right]$$

$$=(6r^3-3r^2)(1-e^{-t})+m_t^{-1}\left[\frac{\Gamma(1.2)r^{1.8}\Gamma(3)r^{0.2}}{\Gamma(1.2)}-\frac{\Gamma(1.2)r^{1.8}\Gamma(4)r^{1.2}}{\Gamma(2.2)}\right]$$

$$=(6r^3-3r^2)(1-e^{-t})+2r^2t-5r^3t$$

By successive iterations, we get

$$y_2(r,t)=(30r^3-6r^2)(t+e^{-t}-1)+(4r^2-25r^3)\frac{t^2}{2}$$

$$y_3(r,t)=(150r^3-12r^2)\left(\frac{t^2}{2}-t-e^{-t}+1\right)+\left(\frac{8r^2-125r^3}{6}\right)t^3$$

3.3 COMPARISON OF ANALYTICAL AND MODIFIED DECOMPOSITION METHOD

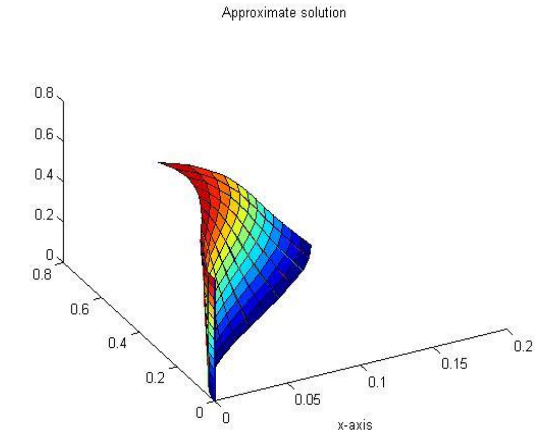

Figure 8: Solution by Modified Decomposition Method

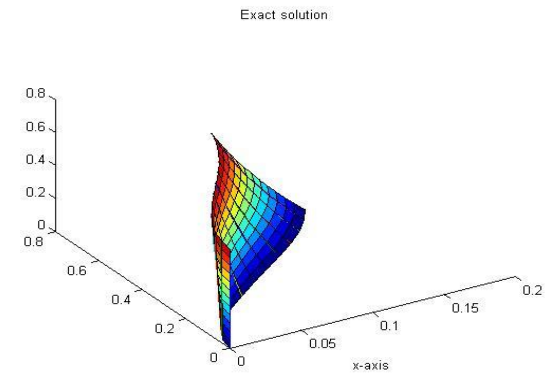

Figure 9: Exact solution

Table-6 When t = 0.1

x	t	Approximate	Exact	Error
0.00000	0.10000	0.00000	0.00000	0.00000
0.05000	0.10000	0.00215	0.00215	0.00000
0.10000	0.10000	0.00814	0.00814	0.00000
0.15000	0.10000	0.01731	0.01731	0.00000
0.20000	0.10000	0.02895	0.02895	0.00000
0.25000	0.10000	0.04241	0.04241	0.00000
0.30000	0.10000	0.05700	0.05700	0.00000
0.35000	0.10000	0.07205	0.07205	0.00000
0.40000	0.10000	0.08686	0.08686	0.00000
0.45000	0.10000	0.10078	0.10078	0.00000
0.50000	0.10000	0.11310	0.11310	0.00000

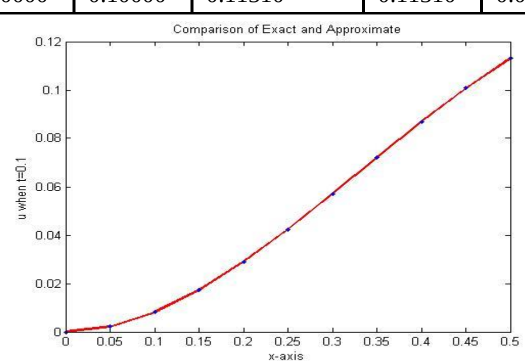

Figure-10 When t = 0.1

Table 7: When t = 0.15

x	t	Approximate	Exact	Error
0.00000	0.15000	0.00000	0.00000	0.00000
0.05000	0.15000	0.00204	0.00204	0.00000
0.10000	0.15000	0.00775	0.00775	0.00000
0.15000	0.15000	0.01646	0.01646	0.00000
0.20000	0.15000	0.02754	0.02754	0.00000
0.25000	0.15000	0.04034	0.04035	0.00000
0.30000	0.15000	0.05422	0.05422	0.00000
0.35000	0.15000	0.06853	0.06853	0.00000
0.40000	0.15000	0.08262	0.08263	0.00000
0.45000	0.15000	0.09586	0.09586	0.00000
0.50000	0.15000	0.10758	0.10759	0.00001

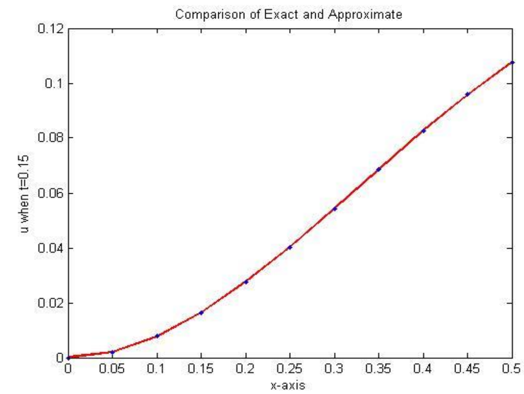

Figure-11 When t = 0.15

Table-8 When t = 0.2

x	t	Approximate	Exact	Error
0.00000	0.20000	0.00000	0.00000	0.00000
0.05000	0.20000	0.00194	0.00194	0.00000
0.10000	0.20000	0.00737	0.00737	0.00000
0.15000	0.20000	0.01566	0.01566	0.00000
0.20000	0.20000	0.02620	0.02620	0.00000
0.25000	0.20000	0.03837	0.03838	0.00000
0.30000	0.20000	0.05157	0.05158	0.00001
0.35000	0.20000	0.06518	0.06519	0.00001
0.40000	0.20000	0.07858	0.07860	0.00002
0.45000	0.20000	0.09116	0.09119	0.00002
0.50000	0.20000	0.10231	0.10234	0.00003

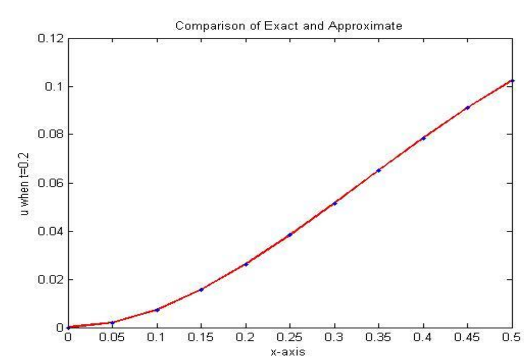

Figure-12 When t = 0.2

Table-9 When t = 0.25

x	T	Approximate	Exact	Error
0.00000	0.25000	0.00000	0.00000	0.00000
0.05000	0.25000	0.00185	0.00185	0.00000
0.10000	0.25000	0.00701	0.00701	0.00000
0.15000	0.25000	0.01489	0.01489	0.00000
0.20000	0.25000	0.02491	0.02492	0.00001
0.25000	0.25000	0.03649	0.03651	0.00002
0.30000	0.25000	0.04904	0.04906	0.00003
0.35000	0.25000	0.06197	0.06201	0.00004
0.40000	0.25000	0.07470	0.07476	0.00006
0.45000	0.25000	0.08665	0.08674	0.00009
0.50000	0.25000	0.09722	0.09735	0.00013

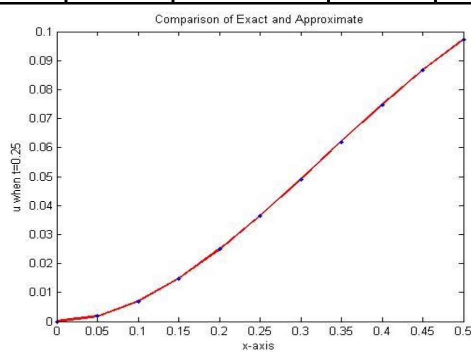

Figure-13 When t = 0.25

Table-10 When t = 0.3

x	t	Approximate	Exact	Error
0.00000	0.30000	0.00000	0.00000	0.00000
0.05000	0.30000	0.00176	0.00176	0.00000
0.10000	0.30000	0.00666	0.00667	0.00000
0.15000	0.30000	0.01416	0.01417	0.00001
0.20000	0.30000	0.02368	0.02371	0.00002
0.25000	0.30000	0.03468	0.03473	0.00005
0.30000	0.30000	0.04659	0.04667	0.00008
0.35000	0.30000	0.05886	0.05899	0.00013
0.40000	0.30000	0.07093	0.07112	0.00019
0.45000	0.30000	0.08224	0.08251	0.00027
0.50000	0.30000	0.09223	0.09260	0.00037

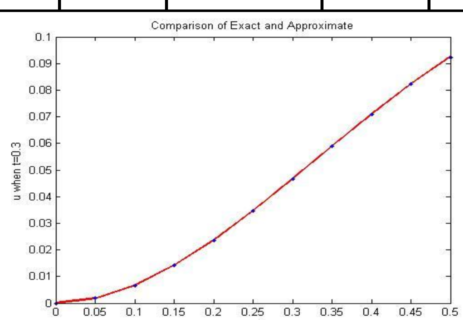

Figure-14 When t = 0

2. CONCLUSION

In above examples we have analyze the results of one dimensional space fractional diffusion equation with source term by analytical and approximate method. The approximate method used was "Modified Decomposition Method (MDM)". We have presented a brief description of MDM. We have used "Mat lab" to get the numerical and graphical results. The results show the validity and potential of MDM. The comparisons of graphical and numerical results show the utility of MDM due to minimum error and quick convergence.

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